MATH 2230 Complex Variables with Applications
Tutorial 9 and 10

The shortest path between two truths in the real domain passes through the complex domain.

– Jacques Hadamard

Please don't give up and give your best! ᘎωᘎ

All the contours in this note are positively oriented. Please let me know if you find any mistake.

1 Complex Differentiable Functions as Analytic Functions

For both real and complex functions, one can define **Analytic Functions**.

Definition. Let Ω be a open subset of $\mathbb{R}$ or $\mathbb{C}$. A function $f : \Omega \to \mathbb{R}$ or $f : \Omega \to \mathbb{C}$ is said to be analytic at $z \in \Omega$ if there exists an open set $D \subseteq \Omega$ and $(c_n) \subset \mathbb{R}$ or $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_k (z - z_0)^n \text{ on } D.$$

f is said to be analytic on Ω if f is analytic everywhere.

From its definition, it is clear that an analytic function is necessarily infinitely differentiable; however, we have

Example 1.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Clearly, f is infinitely differentiable for $x \neq 0$. Moreover, one can show that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. However, f is not analytic at 0; otherwise, f has to be entirely zero in a neighborhood of 0 because of the power series expansion at 0.

However, as we have learnt, given a complex differentiable f on $D(z_0, \delta)$, we have

Theorem. For $0 < r < \delta$,

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \frac{(z - z_0)^{n+1}}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^{n+1}} d\zeta, z \in D(z_0, \delta).$$

Moreover,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, z \in D(z_0, \delta)$$

and the convergence is uniform on $\overline{D(z_0, r)}$ for any $0 < r < \delta$.

As an immediate corollary, we have

Corollary. Let $\Omega \subseteq \mathbb{C}$ be open and f is complex differentiable on Ω, then f is analytic on Ω.

2 Zeros of Analytic Functions

Given an analytic function f, because of the power series expansion, if $f(z_0) = 0$ and f does not vanish on a neighborhood at a point z_0, then z_0 is an **isolated zero** of f, meaning these exist a neighborhood $\mathcal{N}$ of z_0 such that $f(z) \neq 0$ on $\mathcal{N} \setminus \{z_0\}$. Indeed, because of the power series expansion, we have

Theorem. Suppose $\Omega \subseteq \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic, $z_0 \in \Omega$ and $f^{(k)}(z_0) = 0$ for $k = 0, 1, \cdots, n-1$, then there exist an analytic function $\varphi : \Omega \to \mathbb{C}$ such that $\varphi(z) \neq 0$ on a neighborhood $\mathcal{N}$ of z_0 and

$$f(z) = (z - z_0)^n \varphi(z).$$

An immediate corollary is the zeroes of f are isolated

Corollary. Suppose $f : \Omega \to \mathbb{C}$ is analytic and there exists a sequence $(z_n) \subset \Omega$ such that $z_n \to z_0$ as $n \to \infty$ and $f(z_n) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, then there exists a neighborhood $\mathcal{N}$ of z_0 such that $f(z) = 0$ on $\mathcal{N}$.

Some applications of the above facts are

Example 2.1. Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire and $z_0 \in \mathbb{C}$. Suppose $f^{(k)}(z_0) = g^{(k)}(z_0) = 0$ for $k = 0, 1, \cdots, n-1$, show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}.$$

Proof. The above corollary gives φ_1, φ_2 such that $\varphi_1(z), \varphi_2(z) \neq 0$ on $\mathcal{N}_1, \mathcal{N}_2$ respectively and

$$f(z) = (z - z_0)^n \varphi_1(z) \text{ and } g(z) = (z - z_0)^n \varphi_2(z).$$

For $z \notin \mathcal{N}_2 \setminus \{z_0\}$,

$$\frac{f(z)}{g(z)} = \frac{(z - z_0)^n \varphi_1(z)}{(z - z_0)^n \varphi_2(z)} = \frac{\varphi_1(z)}{\varphi_2(z)}.$$

Note that $\varphi_1(z_0) = \frac{f^{(n)}(z_0)}{n!}$ and $\varphi_2(z_0) = \frac{g^{(n)}(z_0)}{n!}$ and φ_1, φ_2 are continuous, so

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\varphi_1(z)}{\varphi_2(z)} = \frac{\varphi_1(z_0)}{\varphi_2(z_0)} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}.$$

□

Example 2.2. Prove that there is no analytic function f in $|z| < 1$ such that

$$f(\frac{1}{n}) = \frac{(-1)^n}{n^2}, n \in \mathbb{N}.$$

Proof. On the contrary, suppose such function f exists. The continuity of f at 0 gives

$$f(0) = \lim_{n \to \infty} f(\frac{1}{n}) = 0.$$

Moreover, the differentiability of f at 0 gives

$$f'(0) = \lim_{n \to \infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0} = \lim_{n \to \infty} \frac{(-1)^n}{n} = 0.$$

By the above corollary, there exists an analytic φ such that $f(z) = z^2 \varphi(z)$. Substituting $z = \frac{1}{n}$ gives

$$\varphi(\frac{1}{n}) = (-1)^n$$

which gives

$$0 = \lim_{n \to \infty} \varphi(\frac{1}{n}) = \lim_{n \to \infty} (-1)^n$$

since φ is continuous at 0, a contradiction. □

Example 2.3. Prove that there is no analytic function f in $|z| < 1$ such that

$$f(\frac{1}{n}) = \frac{1}{2^n}, n \in \mathbb{N}.$$

Proof. On the contrary, suppose such function f exists. The continuity of f at 0 gives

$$f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = 0.$$

By the above corollary, there exists an analytic g_1 such that $f(z) = zg_1(z)$. Substituting $z = \frac{1}{n}$ gives

$$g_1(\frac{1}{n}) = \frac{n}{2^n}.$$

Since

$$\lim_{n\to\infty} g_1(\frac{1}{n}) = 0,$$

we can obtain another analytic g_2 such that $g_1(z) = zg_2(z)$ and

$$g_2(\frac{1}{n}) = \frac{n^2}{2^n}.$$

Inductively, for all $m \in \mathbb{N}$, there exists an analytic g_m such that $f(z) = z^m g(z)$ and

$$g_m(\frac{1}{n}) = \frac{m^2}{2^n},$$

showing that

$$f^{(m)}(0) = 0$$

for any $m \in \mathbb{N}$. However, since f is analytic at 0, $f(z) = 0$ on a neighborhood of 0, contradicting $f(\frac{1}{n}) = \frac{1}{2^n} \neq 0$ for any $n \in \mathbb{N}$. □

3 Maximum Modulus Principle

Recall an important theorem concerning complex differentiable functions

Theorem. Suppose $\Omega \subseteq \mathbb{C}$ is a domain. Let $f : \Omega \to \mathbb{C}$ is complex differentiable. Then there does not exist $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)| \text{ on } \Omega.$$

An application is the following

Example 3.1. Suppose f is a nonconstant analytic function on $|z| \leq 1$ such that $|f(z)|$ is a constant on $|z| = 1$. Show that f has a zero in $|z| < 1$.

Proof. On the contrary, suppose f has no zero in $|z| < 1$. Let $c = |f(z_0)|$ for some $|z_0| = 1$. If $c = 0$, the maximum modulus principle gives

$$|f(z)| \leq c = 0$$

and hence $f = 0$ on $|z| \leq 1$, contracting f is nonconstant on $|z| \leq 1$. If $c > 0$, then $g = \frac{1}{f}$ is well defined and analytic on $|z| \leq 1$ since g has no zero on $|z| \leq 1$. Applying maximum modulus principle to f and g, we get

$$|f(z)| \leq c \text{ and } \frac{1}{|f(z)|} \leq \frac{1}{c}$$

on $|z| \leq 1$. Hence, $|f(z)| = c$ on $|z| \leq 1$. Using maximum modulus principle again, f is a constant on $|z| < 1$, which again contradicts f is nonconstant on $|z| \leq 1$. □

4 Argument Principle and Rouche's Theorem

Note that

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{z}=1 \text{ and } \int_{|z|=1}f(z)dz=0$$

for any analytic functions f on $|z|\leq 1$. It motivates us to consider

Theorem. Let $\Omega\subseteq\mathbb{C}$ be open and $f:\Omega\to\mathbb{C}$ be analytic on Ω with zeroes $z_1,z_2,\cdots,z_n$ in Ω with multiplicities $m_1,m_2,\cdots,m_n$. Let $\gamma\subset\Omega$ be a closed contour and $f(z)\neq 0\ \forall z\in\gamma$, then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}dz=\sum_{k=1}^n m_k n(z_k,\gamma),$$

where $n(z_k,\gamma),k=1,2,\cdots,n$ are the winding numbers of $z_k,k=1,2,\cdots,n$ with respect to γ.

The immediate corollaries are

Corollary. Let $\Omega\subseteq\mathbb{C}$ be open and $f:\Omega\to\mathbb{C}$ be analytic on Ω with zeroes $z_1,z_2,\cdots,z_n$ in Ω with multiplicities $m_1,m_2,\cdots,m_n$. Suppose $\gamma\subset\Omega$ is a simple closed contour, $z_1,z_2,\cdots,z_n$ are all inside the bounded region R enclosed by γ and $f(z)\neq 0\ \forall z\in\gamma$, then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}dz=\sum_{k=1}^n m_k=\text{Total number of zeroes of } f \text{ in } R.$$

Corollary. Let $\Omega\subseteq\mathbb{C}$ be open, $f:\Omega\to\mathbb{C}$ be analytic on Ω and $z\in\mathbb{C}$. Suppose $\gamma\subset\Omega$ is a simple closed contour, R is the bounded region enclosed by γ and $f(z)\neq a\ \forall z\in\gamma$, then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)-a}dz=\text{Total number of solutions of } f=a \text{ in } R.$$

A slight generalization is

Theorem. Let $\Omega\subseteq\mathbb{C}$ be open, $F,G:\Omega\to\mathbb{C}$ be analytic on Ω. Suppose $\gamma\subset\Omega$ is a simple closed contour R is the bounded region enclosed by γ, $F(z),G(z)\neq 0\ \forall z\in\gamma$ and $f=\frac{F}{G}$, then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}dz=\text{Total number of zeroes of } F \text{ in } R-\text{Total number of zeroes of } G \text{ in } R.$$

Proof. Let F,G,γ,R as in the theorem. Let $z_1,z_2,\cdots,z_n$ and $z_1',z_2',\cdots,z_m'$ be the zeroes of F and G in R respectively. Factorizing F,G by the theorem in the previous section,

$$F(z)=\varphi_1(z)\prod_{k=1}^n(z-z_k)^{m_k} \text{ and } G(z)=\varphi_2(z)\prod_{k=1}^m(z-z_k')^{m_k'},$$

where m_k,m_k' are the multiplicities of z_k,z_k' respectively and φ_1,φ_2 are analytic on Ω with no zero in R. Hence,

$$\frac{f'(z)}{f(z)}=\sum_{k=1}^n\frac{m_k}{z-z_k}-\sum_{k=1}^m\frac{m_k'}{z-z_k'}+\frac{(\frac{\varphi_1}{\varphi_2})'(z)}{\frac{\varphi_1}{\varphi_2}(z)}.$$

Integrating both sides over γ, the result follows. □

Using the above theorem, **Argument Principle** and **Rouche's Theorem** follows.

Corollary (Argument Principle)**.** Under the same assumptions with $\Gamma=f(\gamma)$, then

$$n(0,\Gamma)=\text{Total number of zeroes of } F \text{ in } R-\text{Total number of zeroes of } G \text{ in } R.$$

Corollary (Rouche's Theorem)**.** Under the same assumptions, if $|F-G|<|F|$ on γ, then

$$\text{Total number of zeroes of } F \text{ in } R = \text{Total number of zeroes of } G \text{ in } R.$$

Remark. Geometrically, Argument Principle tell us that the difference of the number of zeroes of F and G in R is the number of times that Γ circles around the origin, that is

$$\text{Difference in total number of zeroes of } F \text{ and } G \text{ in } R = \left|\frac{\text{The total change in argument}}{2\pi}\right|.$$

Remark. Geometrically, Rouche's Theorem is case where Γ does not circle around the origin at all, so

$$\text{Difference in total number of zeroes of } F \text{ and } G \text{ in } R = 0.$$

In some case, Rouche's Theorem can be useful in determining the solutions of an equation as the following examples illustrate.

Example 4.1. Determine the number of zeroes, including multiplicity, of the following polynomials in $|z|<1$.

1. $z^6-5z^4+z^3-2z$
2. $2z^4-2z^3+2z^2-2z+9$

Solution:

1. Let $g(z)=z^6-5z^4+z^3-2z$ and $f(z)=-5z^4$, then
$$|f-g|\le 1+1+2=4<5=|f|$$
on $|z|=1$. Hence, Rouche's Theorem shows that g has 4 zeroes in $|z|<1$.
2. Let $g(z)=2z^4-2z^3+2z^2-2z+9$ and $f(z)=9$, then
$$|f-g|\le 2+2+2+2=8<9=|f|$$
on $|z|=1$. Hence, Rouche's Theorem shows that g has no zero in $|z|<1$.

Example 4.2. Determine the number of zeroes, including multiplicity, of the following polynomials in $|z|<2$.

1. z^4+3z^3+6
2. $z^4-2z^3+9z^2+z-1$

Solution:

1. Let $g(z)=z^4+3z^3+6$ and $f(z)=3z^3$, then
$$|f-g|\le 16+6=22<24=|f|$$
on $|z|=2$. Hence, Rouche's Theorem shows that g has 3 zeroes in $|z|<2$.
2. $g(z)=z^4-2z^3+9z^2+z-1$ and $f(z)=9z^2$, then
$$|f-g|\le 16+16+2+1=35<36=|f|$$
in $|z|=2$. Hence, Rouche's Theorem shows that g has 2 zeroes in $|z|<2$.

3. Let $g(z) = z^5 + 3z^3 + z^2 + 1$ and $f(z) = z^5$, then

$$|f - g| \le 24 + 4 + 1 = 29 < 32 = |f|$$

on $|z| = 2$. Hence, Rouche's Theorem shows that g has 5 zeroes in $|z| < 2$.

Example 4.3. Determine the number of zeroes, including multiplicity, of the polynomials $2z^5 - 6z^2 + z + 1$ in $1 \le |z| < 2$.
Solution: Let $g(z) = 2z^5 - 6z^2 + z + 1$, $f_1(z) = -6z^2$ and $f_2(z) = 2z^5$, then

$$|f_1 - g| \le 2 + 1 + 1 = 4 < 6 = |f_1|$$

on $|z| = 1$ and

$$|f_2 - g| \le 24 + 2 + 1 = 27 < 64 = |f_2|$$

on $|z| = 2$. Hence, Rouche's Theorem shows that g has 2 zeroes in $|z| < 1$ and g has 5 zeroes in $|z| < 2$ and so g has 3 zeroes in $1 \le |z| < 2$.

Example 4.4. Suppose $c \in \mathbb{C}$ is such that $|c| > e$. Show that the number of solution, including multiplicity, of the equation $e^z = cz^n$ in $|z| < 1$ is n.
Solution: Consider $g(z) = e^z - cz^n$ and $f(z) = -cz^n$, then

$$|f - g| = |e^z| = e^x \le e < |c| = |f|$$

on $|z| = 1$. Hence, Rouche's Theorem shows that g has n zeroes in $|z| < 1$.

Please refer to the exercises and lecture notes for more examples of this type.

Remark. From the examples, if $\gamma = \{z \in \mathbb{C} : |z| = R\}$ for some $R > 0$ and $g(z) = \sum_{k=1}^{\deg(g)} a_k z^k$ is a polynomial, we see that a technique in applying Rouche's Theorem to get the number of zeroes of g is to set f as a polynomial with simple factorization while $|f|$ is large on γ. In particular, it would be easier to begin the test by considering $f(z) = a_m z^m$, where $1 \le m \le \deg(g)$ is such that $|a_m|R^m = \max_{1 \le k \le \deg(g)} |a_k|R^k$. This kind of method can also somewhat be extend to the case where g is not a polynomial by considering $f(z) = cz^k$ for some $k \in \mathbb{N}$ and $c \in \mathbb{C}$.

5 Laurent Series and Residues

In the lectures, we learned that

Theorem. Let $f : D(z_0, R_2)\backslash\overline{D(z_0, R_1)} \to \mathbb{C}, R_1 < R_2$ be analytic and $\gamma \subset D(z_0, R_2)\backslash\overline{D(z_0, R_1)}$ be a simple closed contour, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, R_1 < |z - z_0| < R_2,$$

where

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and the convergence is uniform on $r_1 \le |z - z_0| \le r_2$ for any $R_1 < r_1 < r_2 < R_2$.

Moreover, the Laurent series (and hence the Taylor's series) of a given function is **unique**

Theorem. Under the same assumption, if

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{'} (z - z_0)^n, R_1 < |z - z_0| < R_2,$$

then $c_n = c_n^{'}$ $\forall n \in \mathbb{Z}$.

The Laurent series of a function f being unique is useful in computing the Laurent series and hence the residue of f.

Example 5.1. Find the residues of the following functions at 0.

1. $\frac{1}{z+z^2}$
2. $z \cos \frac{1}{z}$
3. $\frac{z-\sin z}{z}$
4. $\frac{\cot z}{z^4}$
5. $\frac{\sinh z}{z^4(1-z^2)}$

Solution:

1. For $0 < |z| < 1$,
$$\frac{1}{z+z^2} = \frac{1}{z}\frac{1}{1+z} = \frac{1}{z}\sum_{n=0}^{\infty}(-z)^n = \sum_{n=-1}^{\infty}(-1)^{n+1}z^n.$$
By the uniqueness of Laurent series,
$$\mathrm{Res}_{z=0}\,\frac{1}{z+z^2} = 1.$$
2. For $0 < |z| < \infty$,
$$z\cos(\frac{1}{z}) = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n-1}}.$$
By the uniqueness of Laurent series,
$$\mathrm{Res}_{z=0}\,z\cos(\frac{1}{z}) = -\frac{1}{2}.$$
3. For $0 < |z| < \infty$,
$$\frac{z-\sin z}{z} = \frac{z - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}}{z} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}z^{2n}.$$
By the uniqueness of Laurent series,
$$\mathrm{Res}_{z=0}\,\frac{z-\sin z}{z} = 0.$$
4. For $0 < |z| < \infty$,
$$\frac{\cot z}{z^4} = \frac{1}{z^4}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}} = \frac{1}{z^5} - \frac{1}{3}\frac{1}{z^3} - \frac{1}{45}\frac{1}{z} + \cdots.$$
By the uniqueness of Laurent series,
$$\mathrm{Res}_{z=0}\,\frac{\cot z}{z^4} = \frac{-1}{45}.$$
5. For $|z| < 1$,
$$\frac{\sinh z}{1-z^2} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\sum_{n=0}^{\infty}z^{2n} = z + (1+\frac{1}{3!})z^3 + \cdots.$$
By the uniqueness of Laurent series,
$$\mathrm{Res}_{z=0}\,\frac{\sinh z}{z^4(1-z^2)} = 1 + \frac{1}{3!} = \frac{7}{6}.$$

6 Applications of Theory of Residues

Using Cauchy Residue Theorem, a number of types of improper integrals can be found.

6.1 Improper Integrals

Let $f : \mathbb{R} \to \mathbb{R}$ be an Riemann integrable function on $[-R, R]$ for any $R > 0$. By definition, an improper integral of the form

$$\int_0^\infty f(x)dx$$

is defined by

$$\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx$$

provided the limit exists. Another kind of integral of interest to us is those of the form

$$\int_{-\infty}^\infty f(x)dx,$$

which is defined by

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^0 f(x)dx + \lim_{R\to\infty} \int_0^R f(x)dx,$$

provided both limits exists. Note that we do not define the later integral as

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx$$

to distinguish those functions for which

$$\int_{-R}^0 f(x)dx \text{ and } \int_0^R f(x)dx$$

both diverges while

$$\lim_{R\to\infty} \int_{-R}^R f(x)dx$$

exists as a real number. However, to include

$$\lim_{R\to\infty} \int_{-R}^R f(x)dx$$

as part of our study, we call it the **Cauchy Principle Value** of f and denote it by

$$\text{P. V.} \int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx.$$

To summarize our discussion, we define

Definition. The improper integrals

$$\int_{-\infty}^\infty f(x)dx \text{ and P. V.} \int_{-\infty}^\infty f(x)dx$$

are defined by

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^0 f(x)dx + \lim_{R\to\infty} \int_0^R f(x)dx$$

$$\text{P. V.} \int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx.$$

Example 6.1. For both even and odd functions, that is, functions with the property $f(-x) = \pm f(x)$ respectively, the existences of the integrals

$$\lim_{R\to\infty}\int_{-R}^{0} f(x)dx, \lim_{R\to\infty}\int_{0}^{R} f(x)dx$$

coincide because

$$\int_{-R}^{0} f(x)dx = \pm\int_{0}^{R} f(x)dx$$

for every $R > 0$ by a change of variable respectively. Moreover, for even functions, the existence and values of the 2 kinds of improper integral coincide because

$$\begin{aligned}
\text{P. V.}\int_{-\infty}^{\infty} f(x)dx &= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \\
&= 2\lim_{R\to\infty}\int_{0}^{R} f(x)dx \\
&= \lim_{R\to\infty}\int_{-R}^{0} f(x)dx + \lim_{R\to\infty}\int_{0}^{R} f(x)dx \\
&= \int_{-\infty}^{\infty} f(x)dx.
\end{aligned}$$

For odd functions, however, the Cauchy Principle Values always exists and equal to 0 because

$$\text{P. V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}(\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx) = 0$$

while both one sided improper integrals may diverge.

Example 6.2. Note that $f(x) = x, \sin x$ are both odd functions, so

$$\text{P. V.}\int_{-\infty}^{\infty} f(x)dx = 0.$$

However, both integrals

$$\lim_{R\to\infty}\int_{-R}^{0} f(x)dx, \lim_{R\to\infty}\int_{0}^{R} f(x)dx$$

diverges. For $f(x) = x$, the one-sided improper integrals diverges to $\pm\infty$ while for $f(x) = \sin x$, both one-sided improper integrals simply do not exist at all.

6.2 Improper Integral Type I

In this section, we investigate integrals of the form

$$\text{P. V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = \text{P. V.}\int_{-\infty}^{\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}dx,$$

where no zeroes of Q are real. To ensure the convergence of the integral, we assume $\deg(Q) \geq \deg(P) + 2$. Below is the general method

Theorem. Let P, Q be polynomials with $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0\ \forall x \in \mathbb{R}$. Let $\{z_1, z_2, \cdots, z_N\}$ be all the zeroes of Q with $\operatorname{Im} z_k > 0\ \forall 1 \leq k \leq N$, then

$$\text{P. V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i\sum_{k=1}^{N} \operatorname{Res}(\frac{P}{Q}; z_k).$$

Proof. Let C_R be the upper-semi circle with radius R and γ_R be C_R together with the line segment from $-R$ to R. For $R > \max_{1\leq k\leq N}|z_k|$, by Residue Theory,

$$\int_{\gamma_R} \frac{P(z)}{Q(z)}dz = 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}; z_k).$$

For sufficiently large R,

$$|\int_{C_R} \frac{P(z)}{Q(z)}dz| \leq \pi R \frac{\max_{z\in C_R}|P(z)|}{\min_{z\in C_R}|Q(z)|} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

since $\deg(Q) \geq \deg(P) + 2$. Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}; z_k) - \lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}dz = 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}; z_k).$$

□

Example 6.3. Find

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}.$$

Solution: Let C_R be the upper-semi circle with radius R and γ_R be C_R together with the line segment from $-R$ to R. The zeroes of $(z^2+1)(z^2+4)$ is $\pm i, \pm 2i$. For $R > 2$, by Residue Theory,

$$\int_{\gamma_R} \frac{z^2 dz}{(z^2+1)(z^2+4)} = 2\pi i(\frac{i^2}{(i+i)(i^2+4)} + \frac{(2i)^2}{((2i)^2+1)(2i+2i)}) = \frac{\pi}{3}.$$

Moreover, for $R > 2$,

$$|\int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)}| \leq \pi R \frac{R^2}{(R^2-1)(R^2-4)} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

Therefore,

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{1}{2}(\frac{\pi}{3} - \lim_{R\to\infty}\int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)}) = \frac{\pi}{6}.$$

6.3 Improper Integral Type II

In this section, we investigate integrals of the form

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x dx, \text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x dx,$$

where P, Q are polynomials, $0 \neq \alpha \in \mathbb{R}$ and no zeroes of Q are real. However, unlike the last section where we assume $\deg(Q) \geq \deg(P) + 2$, we can release the assumption to $\deg(Q) \geq \deg(P) + 1$ because of the oscillations of $\sin \alpha x, \cos \alpha x$. We need a inequality called **Jordan's inequality**

Lemma 6.4 (Jordan's inequality)**.** Let $R > 0$, then

$$\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R}.$$

Remark. You may use this inequality without proof in the examination if needed.

Below is the general method

Theorem. Let P, Q be polynomials with $\deg(Q) \geq \deg(P)+1$ and $Q(x) \neq 0\ \forall x \in \mathbb{R}$. Let $\{z_1, z_2, \cdots, z_N\}$ be all the zeroes of Q with $\operatorname{Im} z_k > 0\ \forall 1 \leq k \leq N$, then

$$\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x dx + i\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x dx = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(\frac{P}{Q} e^{(i\alpha\cdot)}; z_k).$$

Proof. Let C_R be the upper-semi circle with radius R and γ_R be C_R together with the line segment from $-R$ to R. For $R > \max_{1\leq k\leq N} |z_k|$, by Residue Theory,

$$\int_{\gamma_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(\frac{P}{Q} e^{(i\alpha\cdot)}; z_k).$$

For sufficiently large R, by Jordan's inequality,

$$\begin{aligned}
|\int_{C_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz| &= |\int_0^{\pi} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})} e^{i\alpha Re^{i\theta}} iRe^{i\theta} d\theta| \\
&\leq R \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \int_0^{\pi} |e^{i\alpha Re^{i\theta}}| d\theta \\
&= R \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \int_0^{\pi} e^{-\alpha R \sin\theta} d\theta \\
&< R \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \frac{\pi}{aR} \\
&= \frac{\pi}{a} \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \longrightarrow 0 \text{ as } R \longrightarrow \infty.
\end{aligned}$$

since $\deg(Q) \geq \deg(P)+1$. Therefore,

$$\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(\frac{P}{Q} e^{(i\alpha\cdot)}; z_k) - \lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(\frac{P}{Q} e^{(i\alpha\cdot)}; z_k).$$

□

Example 6.5. Find

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)},$$

where $a > b > 0$.

Solution: Let C_R be the upper-semi circle with radius R and γ_R be C_R together with the line segment from $-R$ to R. The zeroes of $(z^2+a^2)(z^2+b^2)$ is $\pm ai, \pm bi$. For $R > \max\{a, b\}$, by Residue Theory,

$$\int_{\gamma_R} \frac{e^{iz} dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i (\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}) = \frac{\pi}{a^2-b^2} (\frac{e^{-b}}{b} - \frac{e^{-a}}{a}).$$

Moreover, for $R > \max\{a, b\}$,

$$\begin{aligned}
|\int_{C_R} \frac{e^{iz} dz}{(z^2+a^2)(z^2+b^2)}| &\leq \int_{C_R} \frac{|e^{iz}|}{|(z^2+a^2)(z^2+b^2)|} |dz| \\
&= \int_{C_R} \frac{e^{-y}}{|(z^2+a^2)(z^2+b^2)|} |dz| \\
&\leq \int_{C_R} \frac{1}{(R^2-a^2)(R^2-b^2)} |dz| \ (y \geq 0 \Rightarrow e^{-y} \leq 1) \\
&= \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \longrightarrow 0 \text{ as } R \longrightarrow \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} &= \operatorname{Re} \frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}) - \lim_{R\to\infty} \operatorname{Re} \int_{C_R} \frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)} \\ &= \frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}).\end{aligned}$$

Remark. Sometimes, it is not necessary to use Jordan's inequality to conclude the result because the difference in degree is large enough as the above example shows. However, it is not always the case as the following example shows.

Example 6.6. Find

$$\int_0^{\infty} \frac{x \sin 2x}{x^2+3} dx.$$

Solution: Let C_R be the upper-semi circle with radius R and γ_R be C_R together with the line segment from $-R$ to R. The zeroes of z^2+3 is $\pm\sqrt{3}i$. For $R > \sqrt{3}$, by Residue Theory,

$$\int_{\gamma_R} \frac{ze^{2iz}dz}{z^2+3} = 2\pi i \frac{\sqrt{3}ie^{-2\sqrt{3}}}{2\sqrt{3}i} = i\pi e^{-2\sqrt{3}}.$$

Moreover, for $R > \sqrt{3}$, by Jordan's inequality,

$$\begin{aligned}|\int_{C_R} \frac{ze^{2iz}dz}{z^2+3}| &= |\int_0^{\pi} \frac{Re^{i\theta}e^{2iRe^{i\theta}}}{(Re^{i\theta})^2+3} Rie^{i\theta}d\theta| \\ &\le \frac{R^2}{R^2-3} \int_0^{\pi} e^{-2R\sin\theta} d\theta \\ &< \frac{R^2}{R^2-3} \frac{\pi}{2R} \longrightarrow 0 \text{ as } R \longrightarrow \infty.\end{aligned}$$

Therefore,

$$\int_0^{\infty} \frac{x \sin 2x}{x^2+3} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3} dx = \frac{1}{2}(\operatorname{Im} i\pi e^{-2\sqrt{3}} - \lim_{R\to\infty} \operatorname{Im} \int_{C_R} \frac{ze^{2iz}dz}{z^2+3}) = \frac{\pi}{2} e^{-2\sqrt{3}}.$$

6.4 Improper Integral Type III

In this section, we investigate integrals of the form

$$\int_0^{\infty} \frac{P(x)}{Q(x)} \ln x dx, \int_0^{\infty} \frac{P(x)}{Q(x)} x^{\alpha} dx,$$

where P, Q are polynomials, $-1 < \alpha < 1$ and no zeroes of Q are real. We assume $\deg(Q) \ge \deg(P) + 2$. Moreover, we assume $P(-x) = P(x)$ and $Q(-x) = Q(x)$ on $\mathbb{R}$. Below is the general method

Theorem. Let P, Q be polynomials with $\deg(Q) \ge \deg(P) + 2$ and $Q(x) \ne 0$ $\forall x \in \mathbb{R}$. Let $\{z_1, z_2, \cdots, z_N\}$ be all the zeroes of Q with $\operatorname{Im} z_k > 0$ $\forall 1 \le k \le N$. In addition, assume $P(-x) = P(x)$ and $Q(-x) = Q(x)$ on $\mathbb{R}$, then for $-1 < \alpha < 1$, $-\frac{\pi}{2} < \arg z < \frac{3\pi}{3}$,

$$\int_0^{\infty} \frac{P(x)}{Q(x)} x^{\alpha} dx = \frac{\pi}{\cos\frac{\pi\alpha}{2}} ie^{-i\frac{\pi\alpha}{2}} \sum_{k=1}^{N} \operatorname{Res}(\frac{P}{Q}(\cdot)^{\alpha}; z_k)$$

and

$$\int_0^{\infty} \frac{P(x)}{Q(x)} \ln x dx = \operatorname{Re} \pi i \sum_{k=1}^{N} \operatorname{Res}(\frac{P}{Q} \log; z_k).$$

Proof. We only consider the case where x^α is present and the other case can be treated similarly. Let $C_R, C_\rho, R > \rho > 0$ be the upper-semi circle with radius R and ρ respectively. Let $L^1_{\rho,R}$ be the line segment from $-R$ to $-\rho$ and $L^2_{\rho,R}$ be the line segment from ρ to R. Finally, let $\gamma_{\rho,R} = C_R + L^1_{\rho,R} - C_\rho + L^2_{\rho,R}$. By Residue Theory, for $\rho < 1 < R$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$,

$$\int_{\gamma_{\rho,R}} \frac{P(z)}{Q(z)} z^\alpha dz = 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k).$$

For $R > 1$, since $\alpha < 1$ and $\deg(Q) \geq \deg(P) + 2$,

$$|\int_{C_R} \frac{P(z)}{Q(z)} z^\alpha dz| \leq \pi R^{\alpha+1} \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

For $\rho < 1$, since $\alpha > -1$,

$$|\int_{C_\rho} \frac{P(z)}{Q(z)} z^\alpha dz| \leq \pi \rho^{\alpha+1} \frac{\max_{z\in C_\rho} |P(z)|}{\min_{z\in C_\rho} |Q(z)|} \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Hence,

$$\begin{aligned}
\int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx + \int_{-\infty}^0 \frac{P(x)}{Q(x)} |x|^\alpha e^{\pi i\alpha} dx &= 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k) \\
(1 + e^{\pi i\alpha}) \int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx &= 2\pi i \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k) \\
\int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx &= \frac{\pi}{\cos\frac{\pi\alpha}{2}} i e^{-i\frac{\pi\alpha}{2}} \sum_{k=1}^{N} \text{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k).
\end{aligned}$$

□

Example 6.7. Find

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2} dx,$$

where $-1 < \alpha < 3$.

Solution: Let $C_R, C_\rho, R > \rho > 0$ be the upper-semi circle with radius R and ρ respectively. Let $L^1_{\rho,R}$ be the line segment from $-R$ to $-\rho$ and $L^2_{\rho,R}$ be the line segment from ρ to R. Finally, let $\gamma_{\rho,R} = C_R + L^1_{\rho,R} - C_\rho + L^2_{\rho,R}$. The zeroes of $(z^2+1)^2$ are $\pm i$. By Residue Theory, for $\rho < 1 < R$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$,

$$\begin{aligned}
\int_{\gamma_{\rho,R}} \frac{z^\alpha}{(z^2+1)^2} dz &= 2\pi i(-2\frac{i^\alpha}{(i+i)^3} + \alpha\frac{i^{\alpha-1}}{(i+i)^2}) \\
&= 2\pi i(\frac{e^{i\frac{\alpha\pi}{2}}}{4i} - \alpha\frac{e^{i\frac{(\alpha-1)\pi}{2}}}{4}) \\
&= \frac{\pi}{2}(1-\alpha)e^{i\frac{\alpha\pi}{2}}.
\end{aligned}$$

For $R > 1$, since $\alpha < 3$,

$$|\int_{C_R} \frac{z^\alpha}{(z^2+1)^2} dz| \leq \frac{\pi R^{\alpha+1}}{(R^2-1)^2} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

For $\rho < 1$, since $\alpha > -1$,

$$|\int_{C_\rho} \frac{z^\alpha}{(z^2+1)^2} dz| \leq \frac{\pi \rho^{\alpha+1}}{(1-\rho^2)^2} \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Hence,

$$\begin{aligned}\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}dx + \int_{-\infty}^0 \frac{|x|^\alpha e^{i\alpha\pi}}{(x^2+1)^2} &= \frac{\pi}{2}(1-\alpha)e^{i\frac{\alpha\pi}{2}}\\ (1+e^{i\alpha\pi})\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}dx &= \frac{\pi}{2}(1-\alpha)e^{i\frac{\alpha\pi}{2}}\\ \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}dx &= \frac{\pi(1-\alpha)}{4\cos\frac{\alpha\pi}{2}}.\end{aligned}$$

6.5 Improper Integral Type IV

So far, we have only considered $Q(x) \neq 0 \ \forall x \in \mathbb{R}$ because we want to integrate over the upper semi-circle together with 2 real line segments which eventually covers the whole real line. In this section, we are going to slightly remove this limitation and investigate integrals of the form

$$\int_0^\infty \frac{P(x)}{Q(x)}\ln x dx, \int_0^\infty \frac{P(x)}{Q(x)}x^\alpha dx,$$

where P, Q are polynomials, $-1 < \alpha < 1$ and the zeroes of Q have negative real parts, that is, $\operatorname{Re} z < 0$. We assume $\deg(Q) \geq \deg(P) + 2$. We need a proposition

Proposition. Let g be an entire function with the real part of zeroes being negative and let $f(x) = x^\alpha, \alpha \in \mathbb{R}$ or $f(x) = \ln x$, where $x > 0$. Let $0 < \rho < R$ such that all the zeroes of g are lying in $\rho < |z| < R$, then

$$\int_{|z|=R} \frac{f(z)}{g(z)}dz + \int_{-(|z|=\rho)} \frac{f(z)}{g(z)}dz + \int_\rho^R \frac{f(x)}{g(x)}dx + \int_R^\rho \frac{h(x)}{g(x)}dx = 2\pi i \sum_{z\in Z(g)} \operatorname{Res}(\frac{f}{g}; z),$$

where $Z(g) := \{z \in \mathbb{C} : z \text{ is a zero of } g\}$, $f(z)$ is corresponding function with the branch $0 < \arg z < 2\pi$ and $h(x) = x^\alpha e^{2\pi i\alpha}$ if $f(x) = x^\alpha$ and $h(x) = \ln x + 2\pi i$ if $f(x) = \ln x$.

Remark. One can improve this proposition to the following: Let g be an entire function with all its zeroes lying in $|\arg z| < \varepsilon$ for some $\varepsilon > 0$. Under the same further assumptions and notations as above, then

$$\int_{|z|=R} \frac{f(z)}{g(z)}dz + \int_{-(|z|=\rho)} \frac{f(z)}{g(z)}dz + \int_\rho^R \frac{f(x)}{g(x)}dx + \int_R^\rho \frac{h(x)}{g(x)}dx = 2\pi i \sum_{z\in Z(g)} \operatorname{Res}(\frac{f}{g}; z).$$

Proof. We consider the case where $f(x) = x^\alpha$; the case $f(x) = \ln x$ can be treated similarly. Let $\theta \in (\pi, \frac{3\pi}{2})$. Firstly, let $C_R, C_\rho, R > \rho > 0$ be the arc with radius R and ρ and going from 0 to $Re^{i\theta}$ and $\rho e^{i\theta}$ respectively. Let $L^1_{\rho,R}$ be the line segment from $Re^{i\theta}$ to $\rho e^{i\theta}$ and $L^2_{\rho,R}$ be the line segment from ρ to R. Finally, let $\gamma_{\rho,R} = C_R + L^1_{\rho,R} - C_\rho + L^2_{\rho,R}$. Consider the branch $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ and denote $f_1(z) = x^\alpha$ with this branch. By Cauchy Residue Theorem, since all zeroes of g are in $\rho < |z| < R$ and $f_1 = f$ on $\operatorname{Re} z < 0$,

$$\int_{\gamma_{\rho,R}} \frac{f_1(z)}{g(z)}dz = 2\pi i \sum_{z\in Z(g)} \operatorname{Res}(\frac{f_1}{g}; z) = 2\pi i \sum_{z\in Z(g)} \operatorname{Res}(\frac{f}{g}; z).$$

Secondly, let $C'_R, C'_\rho, R > \rho > 0$ be the arc with radius R and ρ and going from $Re^{i\theta}$ and $\rho e^{i\theta}$ to 0 respectively. Let $L^1_{\rho,R}$ be the line segment from $Re^{i\theta}$ to $\rho e^{i\theta}$ and $L^2_{\rho,R}$ be the line segment from ρ to R. Finally, let $\gamma'_{\rho,R} = C'_R - L^1_{\rho,R} - C_\rho - L^2_{\rho,R}$. Consider the branch $\frac{\pi}{2} < \arg z < \frac{5\pi}{2}$ and denote $f_2(z) = x^\alpha$ with this branch. By Cauchy-Goursat Theorem,

$$\int_{\gamma'_{\rho,R}} \frac{f_2(z)}{g(z)}dz = 0.$$

Note that $f_1 = f$ on $[\rho, R]$, $f_2 = h$ on $[\rho, R]$, $f_1 = f_2$ on $L^1_{\rho,R}$, $f_1 = f_2 = f$ on $|z| = \rho, R$ except $z = \rho, R$; therefore,

$$
\begin{aligned}
&2\pi i \sum_{z\in Z(g)} \operatorname{Res}(\frac{f}{g}; z) \\
&= \int_{\gamma_{\rho,R}} \frac{f_1(z)}{g(z)} dz + \int_{\gamma'_{\rho,R}} \frac{f_2(z)}{g(z)} dz \\
&= \int_{C_R + L^1_{\rho,R} - C_\rho + L^2_{\rho,R}} \frac{f_1(z)}{g(z)} dz + \int_{C'_R - L^1_{\rho,R} - C_\rho - L^2_{\rho,R}} \frac{f_2(z)}{g(z)} dz \\
&= (\int_{C_R} \frac{f_1}{g} dz + \int_{C'_R} \frac{f_2}{g} dz) + (\int_{-C_\rho} \frac{f_1}{g} dz + \int_{-C'_\rho} \frac{f_2}{g} dz) \\
&+ (\int_{L^1_{\rho,R}} \frac{f_1}{g} dz + \int_{-L^1_{\rho,R}} \frac{f_2}{g} dz) + (\int_{L^2_{\rho,R}} \frac{f_1}{g} dz + \int_{-L^2_{\rho,R}} \frac{f_2}{g} dz) \\
&= (\int_{C_R} \frac{f}{g} dz + \int_{C'_R} \frac{f}{g} dz) + (\int_{-C_\rho} \frac{f}{g} dz + \int_{-C'_\rho} \frac{f}{g} dz) \\
&+ (\int_{L^1_{\rho,R}} \frac{f_1}{g} dz + \int_{-L^1_{\rho,R}} \frac{f_1}{g} dz) + (\int_{L^2_{\rho,R}} \frac{f}{g} dx + \int_{-L^2_{\rho,R}} \frac{h}{g} dx) \\
&= \int_{|z|=R} \frac{f}{g} dz + \int_{-(|z|=\rho)} \frac{f}{g} dz + \int_\rho^R \frac{f}{g} dx + \int_R^\rho \frac{h}{g} dx.
\end{aligned}
$$

□

Below is the general method

Theorem. Let P, Q be polynomials with $\deg(Q) \geq \deg(P) + 2$ and all the zeroes of Q lie in $\operatorname{Re} z < 0$. Let $\{z_1, z_2, \cdots, z_N\}$ be all the zeroes of Q, then for $-1 < \alpha < 1$, $0 < \arg z < 2\pi$,

$$\int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx = -\frac{\pi}{\sin \pi\alpha} e^{-\pi i \alpha} \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k).$$

Proof. For sufficiently large R,

$$|\int_{C_R} \frac{P(z)}{Q(z)} z^\alpha dz| \leq \pi R^{1+\alpha} \frac{\max_{z\in C_R} |P(z)|}{\min_{z\in C_R} |Q(z)|} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

since $\deg(Q) \geq \deg(P) + 2$. For sufficiently small ρ,

$$|\int_{C_\rho} \frac{P(z)}{Q(z)} z^\alpha dz| \leq \pi \rho^{1+\alpha} \frac{\max_{z\in C_\rho} |P(z)|}{\min_{z\in C_\rho} |Q(z)|} \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Therefore, by modifying the proof of the previous proposition, we get

$$\int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx + \int_\infty^0 \frac{P(x)}{Q(x)} (x^\alpha e^{2\pi i \alpha}) dx = 2\pi i \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k)$$

$$\int_0^\infty \frac{P(x)}{Q(x)} x^\alpha dx = -\frac{\pi}{\sin \pi\alpha} e^{-\pi i \alpha} \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\cdot)^\alpha; z_k).$$

□

Theorem. Let P, Q be polynomials with $\deg(Q) \geq \deg(P) + 2$ and all the zeroes of Q lie in $\operatorname{Re} z < 0$. Let $\{z_1, z_2, \cdots, z_N\}$ be all the zeroes of Q, then for $0 < \arg z < 2\pi$,

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx = -\frac{1}{2} \operatorname{Re} \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\log)^2; z_k).$$

Proof. For sufficiently large R,

$$|\int_{C_R} \frac{P(z)}{Q(z)} (\log z)^2 dz| \leq \pi R((\ln R)^2 + \pi^2) \frac{\max_{z \in C_R} |P(z)|}{\min_{z \in C_R} |Q(z)|} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

since $\deg(Q) \geq \deg(P) + 2$. For sufficiently small ρ,

$$|\int_{C_\rho} \frac{P(z)}{Q(z)} (\log z)^2 dz| \leq \pi \rho((\ln \rho)^2 + \pi^2) \frac{\max_{z \in C_\rho} |P(z)|}{\min_{z \in C_\rho} |Q(z)|} \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Therefore, by modifying the proof of the previous proposition, we get

$$\begin{aligned}
\int_0^\infty \frac{P(x)}{Q(x)} (\ln x)^2 dx + \int_\infty^0 \frac{P(x)}{Q(x)} (\ln x + 2\pi i)^2 dx &= 2\pi i \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\log)^2; z_k) \\
-4\pi i \int_0^\infty \frac{P(x)}{Q(x)} \ln x dx + 4\pi^2 \int_0^\infty \frac{P(x)}{Q(x)} dx &= 2\pi i \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\log)^2; z_k) \\
\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx &= -\frac{1}{2} \operatorname{Re} \sum_{k=1}^N \operatorname{Res}(\frac{P}{Q}(\log)^2; z_k).
\end{aligned}$$

□

Example 6.8. Find

$$\int_0^\infty \frac{dx}{x^p(x+1)},$$

where $0 < p < 1$.

Solution: By the above proposition,

$$\begin{aligned}
&\int_\rho^R \frac{dx}{x^p(x+1)} + \int_R^\rho \frac{dx}{x^p e^{2\pi i p}(x+1)} dx \\
&+ \int_{|z|=R} \frac{z^{-p}}{(z+1)} dz - \int_{|z|=\rho} \frac{z^{-p}}{(z+1)} dz \\
&= 2\pi i (-1)^{-p} \\
&= 2\pi i e^{-\pi i p}.
\end{aligned}$$

For $R > 1$,

$$|\int_{|z|=R} \frac{z^{-p}}{(z+1)} dz| \leq \frac{2\pi R^{1-p}}{R-1} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

For $\rho < 1$,

$$|\int_{|z|=\rho} \frac{z^{-p}}{(z+1)} dz| \leq \frac{2\pi \rho^{1-p}}{(1-\rho)} \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Therefore,

$$\begin{aligned}
(1 - e^{-2\pi i p}) \int_0^\infty \frac{dx}{x^p(x+1)} &= 2\pi i e^{-\pi i p} \\
\int_0^\infty \frac{dx}{x^p(x+1)} &= 2\pi i \frac{1}{e^{\pi i p} - e^{-\pi i p}} \\
\int_0^\infty \frac{dx}{x^p(x+1)} &= \frac{\pi}{\sin p\pi}.
\end{aligned}$$

Remark. If the proof the proposition is needed in the examination, you need to prove it to get the full mark. Of course, you just have to prove this fact for exactly that problem.

6.6 Improper Integral Type V

In this section, we investigate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cdots, \cos n\theta, \sin n\theta) d\theta$$

where R is a rational function. We want to reduce this type of integrals into integral type I. Below is a method

Theorem. Let R be a rational function, then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta, \cdots, \cos n\theta, \sin n\theta) d\theta$$
$$= \int_{|z|=1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}, \cdots, \frac{z^n+z^{-n}}{2}, \frac{z^n-z^{-n}}{2i})\frac{dz}{iz}.$$

Proof. Let $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$and $\cos k\theta = \frac{e^{ik\theta}+e^{-ik\theta}}{2} = \frac{z^k+z^{-k}}{2}$ and $\sin k\theta = \frac{e^{ik\theta}-e^{-ik\theta}}{2i} = \frac{z^k-z^{-k}}{2i}$. Hence,

$$\int_0^{2\pi} R(\cos\theta, \sin\theta, \cdots, \cos n\theta, \sin n\theta) d\theta$$
$$= \int_{|z|=1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}, \cdots, \frac{z^n+z^{-n}}{2}, \frac{z^n-z^{-n}}{2i})\frac{dz}{iz}.$$

□

Example 6.9. Find

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}.$$

Solution: Denote $C = \{z \in \mathbb{C} : |z| = 1\}$.

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \int_C \frac{1}{5+4(\frac{z-z^{-1}}{2i})}\frac{dz}{iz} \\ &= \int_C \frac{dz}{2z^2+5iz-2} \\ &= \int_C \frac{dz}{(z+2i)(z+\frac{i}{2})} \\ &= 2\pi i \frac{1}{-\frac{i}{2i}+2i} \\ &= \frac{4\pi}{3}.\end{aligned}$$

6.7 Improper Integral Type VI

So far we have only considered the case where the zeroes of Q is either non-real or negative. In this section, we investigate integrals of the form

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{f(x)}{Q(x)} dx,$$

where Q is a polynomial which has a simple real zero and f is a suitable function so that the integral make sense (e.g. converge). We need a theorem

Theorem. Let $x_0 \in \mathbb{R}$ and f be analytic on $0 < |z - x_0| < R$. If x_0 is a simple pole of f, then

$$\lim_{\rho \to 0^+} \int_{C_\rho} f(z)dz = \pi i \operatorname{Res}(f; x_0).$$

Proof. Denote $\alpha := \operatorname{Res}(f; x_0)$. Write

$$f(z) = g(z) + \frac{\alpha}{z - x_0}$$

and integrate both sides over C_ρ, then

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + \alpha\pi i.$$

To estimate $\int_{C_\rho} g(z)dz$, noting that g is bounded on $D(x_0, \frac{R}{2})$, say, $|g(z)| \leq M$ on $D(x_0, \frac{R}{2})$, for $\rho < \frac{R}{2}$,

$$|\int_{C_\rho} g(z)dz| \leq M\pi\rho \longrightarrow 0 \text{ as } \rho \longrightarrow 0^+.$$

Therefore,

$$\lim_{\rho \to 0^+} \int_{C_\rho} f(z)dz = \lim_{\rho \to 0^+} (\int_{C_\rho} g(z)dz + \pi i\alpha) = \pi i\alpha.$$

□

Remark. You can try to compare this theorem with the calculation of $\int_0^\infty \frac{\sin x}{x}dx$ in the suggested solution of exercise 10 where I used a direct proof to calculate the term $\lim_{\rho \to 0^+} \int_{C_\rho} \frac{e^{iz}}{z}dz$ there.

Example 6.10. Find

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx.$$

Solution: Considering $\frac{e^{2iz}}{z^2}$ is no good because 0 is not a simple zero of z^2. Instead, we consider

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{1}{2}\int_0^\infty \frac{1 - \cos 2x}{x^2}dx,$$

so that $\frac{1-e^{2iz}}{z}$ is analytic at 0 and 0 is a simple zero of z. Let $C_R, C_\rho, R > \rho > 0$ be the upper-semi circle with radius R and ρ respectively. Let $L^1_{\rho,R}$ be the line segment from $-R$ to $-\rho$ and $L^2_{\rho,R}$ be the line segment from ρ to R. Finally, let $\gamma_{\rho,R} = C_R + L^1_{\rho,R} - C_\rho + L^2_{\rho,R}$. By Cauchy-Goursat Theorem, for any $0 < \rho < R$,

$$\int_{\gamma_{\rho,R}} \frac{1 - e^{2iz}}{z^2}dz = 0.$$

Estimating,

$$|\int_{C_R} \frac{1 - e^{2iz}}{z^2}dz| \leq \int_{C_R} \frac{1 + e^{-2y}}{R^2}|dz| \leq \int_{C_R} \frac{2}{R^2}|dz| = \frac{2\pi}{R} \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

By the above theorem,

$$\lim_{\rho \to 0^+} \int_{C_\rho} \frac{1 - e^{2iz}}{z^2}dz = \pi i \operatorname{Res}(\frac{1 - e^{2iz}}{z^2}; 0) = \pi i \lim_{\rho \to 0^+} \frac{1 - e^{2iz}}{z} = 2\pi.$$

Therefore,

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{1}{2}\int_0^\infty \frac{1 - \cos 2x}{x^2}dx = \frac{1}{4}\int_{-\infty}^\infty \frac{1 - \cos 2x}{x^2}dx = \frac{1}{4}2\pi = \frac{\pi}{2}.$$

Remark. If the proof the theorem is needed in the examination, you need to prove it to get the full mark. Of course, you just have to prove this fact for exactly that problem.